



## NONLINEAR STANDING WAVES IN ELASTIC RESONATORS

Michal Bednařík, Milan Červenka, Petr Koniček

Czech Technical University in Prague, Technická 2, Prague 6, 166 27, Czech Republic  
[bednarik@fel.cvut.cz](mailto:bednarik@fel.cvut.cz)

### ABSTRACT

Nonlinear plane standing waves are investigated in an elastic half-wave cylindrical resonator. It is supposed that a resonator wall yields locally to the inner pressure. Mechanical vibrations of the resonator wall cause both dissipation and dispersion of acoustic waves. The dispersion effects cause that nonlinear interaction are ineffective because the synchronous conditions are not satisfied. For description of nonlinear standing waves is derived the inhomogeneous Korteweg-de Vries-Burgers equation. This equation takes into account thermo-viscous losses of supposed fluids, boundary layer losses, wall losses and dispersion effects caused by both the resonator wall and the acoustic boundary layer. Analyse can be performed by means of numerical computations and approximate analytical solutions.

### 1. INTRODUCTION

Nowadays, we can observe great interest in nonlinear acoustic resonators. This interest is connected with possibilities of using these resonators in various branches of science and technology. Mostly, the authors, which deal with the resonators, suppose that the investigated resonators are hard-walled. This supposition enables us to ignore wall vibrations. However, we cannot assume in many practical cases that resonator walls do not respond to pressure changes inside of resonators. If we take into account wall vibrations of resonators it is possible suppose that resonator walls yields locally to the inner pressure. This resonator behaves as a two-mode system because it is necessary to take into account both a wall resonance and a resonance of a fluid column in a cylindrical resonator. The wall vibrations cause both dispersion and dissipation acoustic effects. The dispersion affects efficiency of the nonlinear acoustic interactions negatively because the synchronous conditions are not satisfied. As the synchronous conditions are not fulfilled the efficiency of energy transfer from the fundamental harmonic is weaker and consequently the cascade processes of generation of higher harmonics are not so significant.

If we intend to describe of nonlinear standing waves in the resonators it is necessary to take into account thermo-viscous losses of supposed fluids and boundary layer effects that cause dissipation and dispersion of acoustic waves. For description of nonlinear standing waves it is possible to derive in the second approximation the inhomogeneous Korteweg-de Vries-Burgers equation. Using of this model equation follows from the idea

that an acoustic field in the resonators can be realized by two counter-propagating plane nonlinear waves.

In this article the method of derivation of the inhomogeneous Korteweg-de Vries-Burgers equation for acoustic waves in elastic wall resonators is outlined.

## 2. MODEL EQUATIONS

For description of the nonlinear plane standing waves in resonator of a constant radius it is possible to use the Kuznetsov's model equation for velocity potential  $\varphi$  in the second approximation (see e.g. [1], [2], [7])

$$\frac{\partial^2 \varphi}{\partial t^2} - c_0^2 \frac{\partial^2 \varphi}{\partial x^2} + \frac{\gamma - 1}{2c_0^2} \frac{\partial}{\partial t} \left( \frac{\partial \varphi}{\partial t} \right)^2 + \frac{\partial}{\partial t} \left( \frac{\partial \varphi}{\partial x} \right)^2 + \frac{\rho_0 K}{S} \frac{\partial^2 \varphi}{\partial t^2} = \frac{b}{2\rho_0 c_0^3} \frac{\partial^3 \varphi}{\partial t^3} - 2B \frac{\partial^{\frac{3}{2}} \varphi}{\partial t^{\frac{3}{2}}}, \quad (1)$$

where  $x$  is space coordinate in the direction of the resonator axis,  $t$  is time,  $c_0^2$  is the small signal sound speed,  $\rho_0$  is the ambient density of the fluid,  $\gamma = c_p / c_v$  is Poisson's number and  $c_p$ , or  $c_v$  is the specific heat under constant pressure, or volume,  $S$  is an inner cross-section of a resonator,  $K = dS/dp$ ,  $p$  is the inner pressure,  $b = \rho_0 \delta$ , where  $\delta$  is the diffusivity of sound and  $B$  is the coefficient which represents boundary layer effects (see e.g [3], [5]).

In this resonator we can imagine the sound field as a superposition of simple waves propagating in opposite directions which are assumed to not interact in the volume of the resonator and they are coupled only by the conditions on the walls of resonator, see [1], [4]. The next possible simplification is when we neglect the fact that the driving piston is moving and thus the position of the boundary of the resonator is unvarying with the time. This assumption is acceptable for very small amplitude of driving piston. With the above mentioned suppositions we can find the solution of this equation in the following form

$$\varphi = \left[ \mu \varphi_+ \left( \mu x, \mu t, \tau_+ = t - \frac{x}{c_0} \right) - \mu \varphi_- \left( \mu x, \mu t, \tau_- = t - \frac{x}{c_0} \right) \right], \quad (2)$$

where  $\mu$  is a small parameter.

Substituting the expression (2) into Eq. (1) and neglecting the terms of the order three and higher and supposing that counter-propagating waves do not interact we can get, likewise [1], [2], [4], the following two equations

$$\frac{\partial v_+}{\partial x} + \frac{1}{c_0} \frac{\partial v_+}{\partial t} - \frac{\beta}{c_0^2} v_+ \frac{\partial v_+}{\partial \tau_+} + \frac{\rho_0 c_0 K}{2S} \frac{\partial v_+}{\partial \tau_+} = \frac{b}{2\rho_0 c_0^3} \frac{\partial^2 v_+}{\partial \tau_+^2} - B \frac{\partial^{\frac{1}{2}} v_+}{\partial \tau_+^{\frac{1}{2}}}, \quad (3)$$

$$-\frac{\partial v_-}{\partial x} + \frac{1}{c_0} \frac{\partial v_-}{\partial t} - \frac{\beta}{c_0^2} v_- \frac{\partial v_-}{\partial \tau_-} + \frac{\rho_0 c_0 K}{2S} \frac{\partial v_-}{\partial \tau_-} = \frac{b}{2\rho_0 c_0^3} \frac{\partial^2 v_-}{\partial \tau_-^2} - B \frac{\partial^{\frac{1}{2}} v_-}{\partial \tau_-^{\frac{1}{2}}}, \quad (4)$$

where  $\beta = \gamma + 1/2$  is the parameter of nonlinearity. We can write for an acoustic velocity

$$v = v_+ - v_-, \quad (5)$$

where  $v_+$  and  $v_-$  are solution of Eqs. (3) and (4).

The length of the resonator of a constant diameter is labelled by  $L$ . It is valid for angular eigenfrequencies that

$$\omega_n = \frac{n\pi c_0}{L}; \quad n = 1, 2, \dots \quad (6)$$

In the case that we consider the harmonic excitation of the standing waves with the piston at the position  $x = L$ , we can express the boundary and initial conditions as follows

$$v = (v_+ - v_-)_{x=L} = v_m \sin(\omega t) \quad (7)$$

and

$$v_{\pm}(t = 0) = 0, \quad (8)$$

where  $v_m$  is an acoustic velocity amplitude of the piston an. We assume that a piston vibrates with the angular frequency  $\omega$ , which is equal to  $2n+1$ -th eigenfrequency of the given resonator, it means that  $\omega = \omega_{2n+1}$ . This assumption causes that higher harmonic components of an acoustic velocity are in coincidence with eigenfrequencies.

Eqs. (3) and (4) together with conditions (7), (8) and (9) can be solved by the method of successive approximation, see [1], [4]. On the basis of this method we obtain these model equations

$$\frac{\partial \bar{v}_{\pm}}{\partial t} - \frac{\beta}{c_0} \bar{v}_{\pm} \frac{\partial \bar{v}_{\pm}}{\partial \tau_{\pm}} + \frac{\rho_0 c_0^2 K}{2S} \frac{\partial \bar{v}_{\pm}}{\partial \tau_{\pm}} - \frac{b}{2\rho_0 c_0^2} \frac{\partial^2 \bar{v}_{\pm}}{\partial \tau_{\pm}^2} + c_0 B \frac{\partial^{\frac{1}{2}} \bar{v}_{\pm}}{\partial \tau_{\pm}^{\frac{1}{2}}} = \frac{v_m c_0}{2L} \sin(\omega \tau). \quad (9)$$

Eqs. (10) represent the inhomogeneous Burgers equation for nonlinear standing waves in elastic resonators and

$$\bar{v}_{\pm}(t, \tau_{\pm}) = v_{\pm}(t, \tau_{\pm}) \pm \frac{v_m x}{2L} \sin(\omega \tau_{\pm}). \quad (10)$$

We can write for the mechanical wall impedance  $Z_m$  that

$$\frac{4\pi}{Z_m} = j\omega \frac{K}{S}, \quad (11)$$

where

$$Z_m = j\omega M_m + \frac{1}{j\omega C_m} + R_m, \quad (1.12)$$

where  $M_m$  is inertance,  $C_m$  is compliance,  $R_m$  mechanical resistance and  $j = \sqrt{-1}$ . The wall resonance frequency is given as

$$\omega_0 = \frac{1}{\sqrt{C_m M_m}}. \quad (13)$$

When  $\omega = \omega_0$  and  $\omega R_m C_m \ll 1$  we can write from Eq. (11) (see e.g. [6]) that

$$\frac{K}{S} = 4\pi C_m \left[ 1 + \left( \frac{\omega}{\omega_0} \right)^2 - j\omega R_m C_m \right]. \quad (14)$$

On the basis of Eq. (14) we can rewrite Eq. (9)

$$\frac{\partial \bar{v}_{\pm}}{\partial t} + \left( 2\pi\rho_0 c_0^2 C_m - \frac{\beta}{c_0} \bar{v}_{\pm} \right) \frac{\partial \bar{v}_{\pm}}{\partial \tau_{\pm}} + \frac{2\pi\rho_0 c_0^2 C_m}{\omega_0^2} \frac{\partial^3 \bar{v}_{\pm}}{\partial \tau_{\pm}^3} - \frac{b + 4\pi\rho_0^2 c_0^4 C_m}{2\rho_0 c_0^2 \omega_0^2} \frac{\partial^2 \bar{v}_{\pm}}{\partial \tau_{\pm}^2} + c_0 B \frac{\partial^{\frac{1}{2}} \bar{v}_{\pm}}{\partial \tau_{\pm}^{\frac{1}{2}}} = \frac{v_m c_0}{2L} \sin(\omega\tau) . \quad (15)$$

Eq. (15) represents the modified inhomogeneous Korteweg-de Vries-Burgers equation.

### 3.CONCLUSIONS

In the presented paper we have derived the modified inhomogeneous Korteweg-de Vries-Burgers equation which enables us to model plane nonlinear standing waves in fluid-filled elastic resonators. The model equation was derived by means of the second order nonlinear theory. Unfortunately, an analytical solution of this equation has not been known. For this reason it is necessary to solve the model equation by a suitable numerical method.

This work was supported by the CTU research program J04/98:212300016.

### REFERENCES

1. M. BEDNARIK, P. KONICEK, Czech. J. of Physics, **54**, 349-355 (2004).
2. M. BEDNARIK, P. KONICEK, J. Acoust. Soc. Am., **115**, 91-98 (2004).
3. M. BEDNARIK, P. KONICEK, J. Acoust. Soc. Am., **112**, 91-98 (2002).
4. V.EM. GUSEV, Sov. Phys. Acoust. **30**, 121-125 (1984).
5. M.F. HAMILTON, D.T. BLACKSTOCK, in Nonlinear Acoustics, 1st edn. (Academic, New York, 1998).
6. T. KAMAKURA, Y. KUMAMOTO, in Proceedings of the 12<sup>th</sup> ISNA, Austin, 1990, edited by M.F. Hamilton and D.T. Blackstock (Elsevier Science Publishers Ltd., London, 1990), p. 333-338.
7. V.P. KUZNETSOV, Sov. Phys. Acoust. **16**, 467-470 (1971).